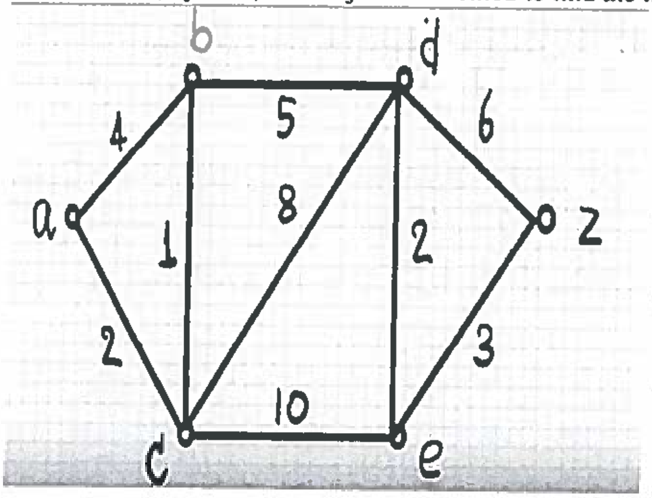
AZADAN BHAGWAGAR 14000

Exam III: MTH 213, Spring 2018

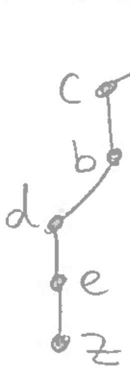
Ayman Badawi

$Score = \frac{37}{37}$

QUESTION 1. (7 points) Use Dijkstra's method to find the minimum spanning tree of the below graph.

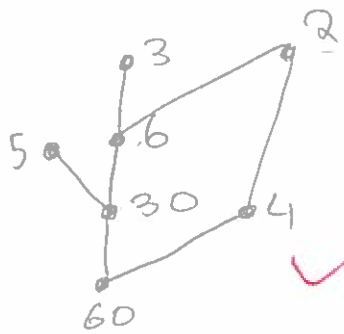


	a	b	c	d	e	z
a	$\boxed{0}$	4^a	2^a	∞	∞	∞
c		3^c	$\boxed{2^a}$	10^c	12^c	∞
b		$\boxed{3^c}$		8^b	12^c	∞
d				$\boxed{8^b}$	10^d	14^d
e					$\boxed{10^d}$	13^e
z						$\boxed{13^e}$



QUESTION 2. $A = \{2, 3, 4, 5, 6, 30, 60\}$. Define $\leq$ on A such that $\forall a, b \in A$, $a \leq b$ if and only if $a = bc$ for some $c \in N^*$. Then $(A, \leq)$ is a partially ordered set (DO NO SHOW THAT)

(i) **(4 points)** Draw the Hassee diagram of such relation



(ii) **(3 points)** By staring at the Hassee diagram, If possible, find

a. $5 \wedge 6$ 30

b. $6 \wedge 4$ 60

c. $6 \vee 3$ 3

d. $30 \vee 60$ 30

e. Is there a $c \in A$ such that $a \leq c$ for every $a \in A$? If yes, find c Does not exist

f. Is there an $m \in A$ such that $m \leq a$ for every $a \in A$? If yes, find m 60

QUESTION 3. (10 points)

(i) Let F be a set with 7 elements, and let $H = \{d \subset F \mid |d| = 4\}$. Find $|H|$ (i.e., find the cardinality of H)

$7C4$

(ii) How many 5-digit even integers greater that 60000 can be formed using the digits (1, 2, 3, 4, 5, 6, 7) such that the second and the third digit must be odd integer.

$2C1$

$\underset{(1^{st})}{1C1} \times \underset{(2^{nd})}{4C1} \times \underset{(3^{rd})}{4C1} \times \underset{(4^{th})}{7C1} \times \underset{(5^{th})}{3C1} = 672$

(iii) There are 7 dots randomly placed on a circle such that exactly 4 of them are red and the remaining three dots are green. How many triangles can be formed within the circle (i.e., inside the circle) such that each triangle has exactly two green vertices?

$3C2 \times 4C1$

(iv) 491 kids are in a gathering, all of them were born between 2010-2014. It is observed that more than 60% of them are girls. Then there exist at least n kids who were born in the same month and in the same year. What is the maximum value of n?

2014-2010+1 = 5

Dom → Codomain

491 → (month, year)

5×12

By Pigeon-hole Principle: $n = \left\lceil \frac{491}{60} \right\rceil = 9$

(v) In the above question, there is a month and a year between 2010-2014 such that at most m kids were born in that month and in that year. Find the minimum value of m.

$m = \left\lfloor \frac{491}{60} \right\rfloor = 8$

QUESTION 4. (4 points)

Given $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Let f be a bijective function from S onto S such that

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 5 & 8 & 6 & 4 & 2 & 3 \end{pmatrix}$$

(i) Find f^2 (i.e., find fof). (Note that by staring at f, we understand that $f(1) = 7, \ldots, f(8) = 3$)

$$f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 7 & 6 & 3 & 4 & 8 & 1 & 5 \end{pmatrix}$$

(ii) Find the least positive integer n such that $f^n = I$, where I is the identity map (i.e., $I(a) = a$ for every $a \in S$)

$f = (1,7,2)(3,5,6,4,8)$; $\therefore n = LCM[3,5] = 15$

QUESTION 5. (2 points) Let $M = Q \cap (-1, 0)$. Is M countable? Is $|M| = |Q|$? explain briefly. Q is countable.

There are infinitely many rational numbers between -1 and 0. $\therefore$ $Q \cap (-1,0)$ is a subset of Q with cardinality infinite countable. So Yes, M is countable, $|M| = |Q|$.

QUESTION 6. **(3 points)** Given $f : [-2, \infty) \to [-4, \infty)$ is a function such that $f(x) = x^3 + \sqrt{x+11} + e^{(x+2)}$. Use mathematical argument and convince me that $\exists! m \in [-2, \infty)$ such that $f(m) = 0$.

$f(-2) = (-2)^3 + \sqrt{9} + e^0 = -4$; $\lim_{x \to \infty} f(x) = \infty + \infty + \infty = \infty$;

range = codomain;

f is continuous, ∴ f is onto.

$f'(x) = 3x^2 + \frac{1}{2(x+11)} + e^{(x+2)}$, $\underset{-2 \qquad\qquad \infty}{+}$, $f'(x)$ is always positive over domain ∴ $f(x)$ is increasing only ∴ $f(x)$ is 1-1.

ok but why!

Since $f(x)$ is onto, there exist 'm' in domain s.t. $f(m) = 0$ because $0 \in [-4, \infty)$, and since $f(x)$ is also 1-1, this m is unique.

QUESTION 7. (4 points)

(a) Let $n = 12.3^2.2^5$. Find $\phi(n)$.

$\phi(n) = \phi(2^2 \times 3 \times 3^2 \times 2^5) = \phi(2^7 \times 3^3)$

$= (1)(2)^6 \times (2)(3)^2 = 1152$

(b) Find 7^{16003} $(mod\, 8)$

$\phi(8) = \phi(2^3) = (1)(2)^2 = 4$; $\gcd(7, 8) = 1$; ∴ By Euler-Fermat result, $7^4 (\text{mod } 8) = 1$; $(7^4)^{4000} \text{ mod } 8 = 1$; $7^{16,000} (\text{mod } 8) = 1$;

∴ $7^{16003} \text{ mod } 8 = 7^3 \times 7^{16000} \text{ mod } 8 = (7^3 \text{ mod } 8) \times (7^{16000} \text{ mod } 8)$

$= 7 \times 1 = 7$

Faculty information

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$((x+1))^{1/2}$

$(1/2)(x+1)^{-1/2}$

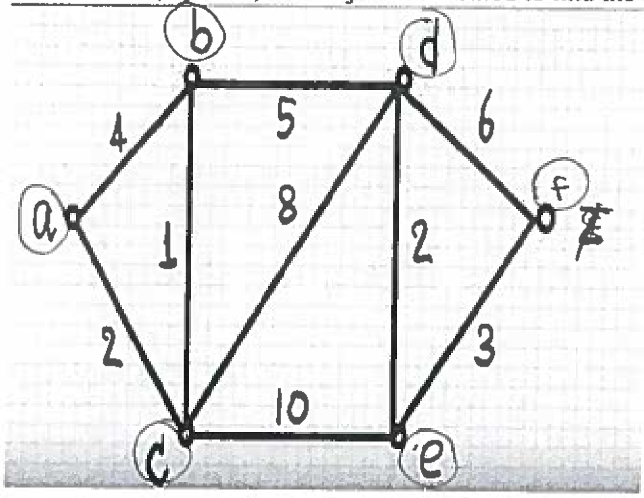
Manam Reda . 75318 .

Exam III: MTH 213, Spring 2018

Ayman Badawi

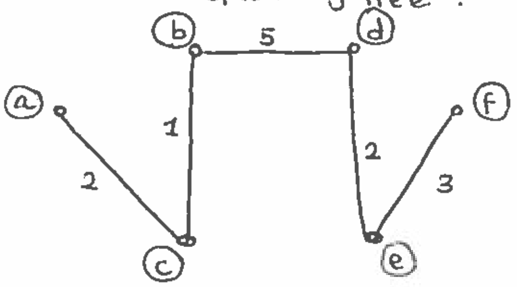
$Score = \frac{36}{37}$

QUESTION 1. (7 points) Use Dijkstra's method to find the minimum spanning tree of the below graph.



	a	b	c	d	e	f
a	$\boxed{0}$	4^a	2^a	∞	∞	∞
c		1+2 3^c	$\boxed{2^a}$	8+2 10^c	10+2 12^c	∞
b		$\boxed{3^c}$		5+3 8^b	12^c	∞
d				$\boxed{8^b}$	2+8 10^d	6+8 14^d
e					$\boxed{10^d}$	3+10 13^e
f						$\boxed{13^e}$

∴ Minimum spanning tree :



QUESTION 2. $A = \{2, 3, 4, 5, 6, 30, 60\}$. Define $\leq$ on A such that $\forall a, b \in A$, $a \leq b$ if and only if $a = bc$ for some $c \in N^*$. Then $(A, \leq)$ is a partially ordered set (DO NO SHOW THAT)

(i) **(4 points)** Draw the Hassee diagram of such relation

~~2,3,4,5,6,30,60.~~

~~...~~ [Not counting itself] .

60 "≤" : 2, 3, 4, 5, 6, 30 .

30 "≤" : 2, 3, 5, 6 .

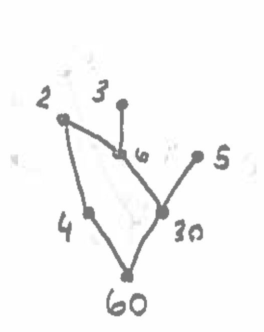
6 "≤" : 2, 3 .

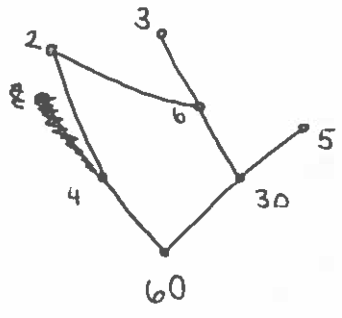
5 "≤" : ϕ, 5 .

4 "≤" : 2

3 "≤" : ϕ, 3 .

2 "≤" : ϕ, 2 .





(ii) **(3 points)** By staring at the Hassee diagram, If possible, find

∧ → greatest lower bound

∨ → least upper bound

a. $5 \wedge 6$ ⇒ x "≤" 5 AND x "≤" 6 (greatest) ⇒ 5∧6 = 30 .

b. $6 \wedge 4$ ⇒ x "≤" 6 AND x "≤" 4 (greatest) ⇒ 6∧4 = 60 .

c. $6 \vee 3$ → 6 "≤" x AND 3 "≤" x (least) ⇒ 6∨3 = 3 .

d. $30 \vee 60$ ⇒ 30 "≤" x AND 60 "≤" x (least) ⇒ 30∨60 = 30 .

e. Is there a $c \in A$ such that $a \leq c$ for every $a \in A$? If yes, find c → No DNE (no single c a "≤" c , 2 ≰ 3 are same).

f. Is there an $m \in A$ such that $m \leq a$ for every $a \in A$? If yes, find m → Yes, m = 60 .

QUESTION 3. (10 points)

(i) Let F be a set with 7 elements, and let $H = \{d \subset F \mid |d| = 4\}$. Find $|H|$ (i.e., find the cardinality of H)

subset

$\therefore |H| = 7C4 = \underline{35}$.

$|F| = 7$.

$|d| = 4$

$\therefore$ Need to find all possible combinations of ~~F~~ elements that make up subsets, order is ~~...~~ not important (for elements in subset)

H has elements where each element is a subset of F s.t. the subset has cardinality 4. $\therefore$

(ii) How many 5-digit even integers greater that 60000 can be formed using the digits (1, 2, 3, 4, 5, 6, 7) such that the second and the third digit must be odd integer.

1^{st} digit : must be 6 or 7 (≥ 60000) $\therefore \boxed{2C1}$.

2^{nd} digit : odd, $\Rightarrow \{1,3,5,7\}$ $\therefore \boxed{4C1}$

3^{rd} digit: odd, $\Rightarrow \{1,3,5,7\}$ $\therefore \boxed{4C1}$.

4^{th} digit : No restriction $\therefore \boxed{7C1}$

5^{th} digit : must be even, $\Rightarrow \{2,4,6\}$ $\therefore \boxed{3C1}$

$\therefore$ Possibilities $= 2C1 \times 4C1 \times 4C1 \times 7C1 \times 3C1$
$= 2 \times 4 \times 4 \times 7 \times 3 = \boxed{672}$.

(iii) There are 7 dots randomly placed on a circle such that exactly 4 of them are red and the remaining three dots are green. How many triangles can be formed within the circle (i.e., inside the circle) such that each triangle has exactly two green vertices?

4 red dots.

3 green dots $\Rightarrow$ must have 2.

Triangle formed = 3 dots. $\therefore$ Possibilities $= 3C2 \times 4C1 = \underline{12}$.

(iv) 491 kids are in a gathering, all of them were born between 2010-2014. It is observed that more than 60% of them (unimportant) are girls. Then there exist at least n kids who were born in the same month and in the same year. What is the maximum value of n?

Domain = Kids. $|\text{Domain}| = 491$.

Co-domain = Month, Year. $|\text{Co-domain}| = 60$.

No. of years $= 2014 - 2010 + 1 = 5$.

No. of months $= 12$.

$\therefore |\text{Codomain}| = |(\text{month}, \text{year})| = 12 \times 5 = 60$.

$\therefore$ max. value of $n = \left\lceil \frac{491}{60} \right\rceil = \lceil 8.18\ldots \rceil = \underline{9}$.

$\Rightarrow$ There are at least 9 kids born in the same month and in the same year.

(v) In the above question, there is a month and a year between 2010-2014 such that at most m kids were born in that month and in that year. Find the minimum value of m.

min. value of $m = \left\lfloor \frac{491}{60} \right\rfloor = \lfloor 8.18\ldots \rfloor = \underline{8}$.

$\Rightarrow$ There are at most 8 kids born in the same month and in the same year.

QUESTION 4. (4 points)

Given $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Let f be a bijective function from S onto S such that

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 5 & 8 & 6 & 4 & 2 & 3 \end{pmatrix}$$

(i) Find f^2 (i.e., find $f o f$). (Note that by staring at f, we understand that $f(1) = 7, \ldots, f(8) = 3$)

$f^2 = f o f = f(f(x))$.

$f(f(1)) = f(7) = 2$.
$f(f(2)) = f(1) = 7$.
$f(f(3)) = f(5) = 6$.
$\vdots$
$f(f(8)) = f(3) = 5$.

$\therefore$ Range of $f^2 = (2\ 7\ 6\ 3\ 4\ 8\ 1\ 5)$

$$\therefore f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 7 & 6 & 3 & 4 & 8 & 1 & 5 \end{pmatrix}$$

(ii) Find the least positive integer n such that $f^n = I$, where I is the identity map (i.e., $I(a) = a$ for every $a \in S$)

Composition of f as disjoint cycles $= \underbrace{(1\ 7\ 2)}_{\text{3-cycle}}\underbrace{(3\ 5\ 6\ 4\ 8)}_{\text{5-cycle}}$.

$\therefore n = \text{Lcm}[3, 5] = \underline{15}$. $\Rightarrow f^{15} = I$.

QUESTION 5. (2 points) Let $M = Q \cap (-1, 0)$. Is M countable? Is $|M| = |Q|$? explain briefly.

If $m = Q \cap (-1,0)$, this means that m is a subset of Q.

Since we know Q is ~~rational~~ countable (it is an infinite set of rational numbers), we can conclude that M is also countable as a subset of a countable set is countable. Where $|Q| = |N^*|$

[since (-1,0) is countable & Q is countable, and countable set $\cap$ countable set = countable set].
(infinitely many rational numbers exist between -1 and 0).

To have same cardinalities, there must exist a bijective function from $M \rightarrow Q$, which is clear since m is a subset of Q. Hence $|M| = |Q| = |N^*|$.

No $|(-1,0)| = |\mathbb{R}| = \infty$, uncountable

QUESTION 6. (3 points) Given $f : [-2, \infty) \to [-4, \infty)$ is a function such that $f(x) = x^3 + \sqrt{x+11} + e^{(x+2)}$. Use mathematical argument and convince me that $\exists! m \in [-2, \infty)$ such that $f(m) = 0$.

Claim : There exists a unique m in $[-2, \infty)$ such that $f(m) = 0$.

$\Rightarrow 0 = m^3 + \sqrt{m+11} + e^{(m+2)}$

If f is a function (as given), this means that :

(1.) $\forall a \in$ Domain, $f(a) \in$ Codomain.

ie. $\forall a \in [-2, \infty)$, $f(a) \in [-4, \infty)$.

and

(2.) $\forall a \in$ Domain, $\{f(a)\}$ has only one element.

ie. $\forall a \in [-2, \infty)$, $f(a)$ has a single solution between $[-4, \infty)$.

since we know f is a function,

$\Rightarrow$ since $m \in [-2, \infty)$:

$\therefore f(m) \in [-4, \infty)$ [based on (1.)]

and $\therefore \{f(m)\}$ has only one element.

$f(x) = x^3 + \sqrt{x+11} + e^{(x+2)}$

$f'(x) = 3x^2 + \frac{1}{2}(x+11)^{-1/2} + (x+2)e^{x+2} > 0$.

$\therefore f(x)$ is an increasing function.

Hence f is one-to-one.

$\Rightarrow$ Each $a \in [-2, \infty)$ has a unique $f(a) \in [-4, \infty)$

and since $0 \in [-4, \infty)$,

$\Rightarrow$ There is a unique $m \in [-2, \infty)$ for which $f(m) = 0$.

QED

why - onto

QUESTION 7. (4 points)

(a) Let $n = 12.3^2.2^5$. Find $\phi(n)$.

$n = 12 \times 3^2 \times 2^5$

$= 3 \times 2 \times 2 \times 3^2 \times 2^5$

$= 3^3 \times 2^7$.

$\therefore \phi(n) = (3-1)(3^{3-1}) \times (2-1)(2^{7-1})$

$= (2)(3^2) \times (1)(2^6) = \underline{1152}$.

(b) Find 7^{16003} $(mod\ 8)$

[We know from fact that, for $a, n \in \mathbb{N}^*$ where $\gcd(a,n) = 1$, the following $a^{\phi(n)} (\text{mod } n) = 1$ is true.]

Let $a = 7$, $n = 8$.

$\gcd(7, 8) = 1$.

$\Rightarrow 7^{\phi(8)} (\text{mod } 8) = 1$ (from fact).

$8 = 2 \times 2 \times 2 = 2^3$

$\phi(8) = (2-1)(2^{3-1})$

$= (1)(2^2) = 4$.

Hence $7^4 (\text{mod } 8) = 1$.

$16003 = 16000 + 3$

$= (4 \times 4000) + 3$.

$\therefore 7^{16003} (\text{mod } 8) = 7^{16000+3} (\text{mod } 8)$

$= 7^{16000} \cdot 7^3 (\text{mod } 8)$

$= 7^{4(4000)} (\text{mod } 8) \times 7^3 (\text{mod } 8)$

$= 1 \times 343 (\text{mod } 8)$

$343 (\text{mod } 8) = 7$.

$\therefore 7^{16003} (\text{mod } 8) = \underline{\underline{7}}$.

Faculty information

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